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A GLANCE AT SOME FUNDAMENTAL ASPECTS OF MATHEMATICS

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IN a recent book Sir Oliver Lodge has said that "the mathematical ignorance of the average educated person has always been complete and shameless." To those who know how vast a body of human achievements the term mathematics has come to denote, to those who are aware of the immense development of the subject in modern times, and especially to such as understand and value its spiritual significance as manifest in its bearings upon the higher concerns of man, this indictment can not fail to seem a pretty terrible charge.

The charge is a double one: complete ignorance and shamelessness. The two counts, however, are not independent and this fact is a mitigating consideration. If the first count be correct, the second must be so too. For complete ignorance is complete innocence and innocence is never ashamed. Is the first count correct? The answer depends. For what does Sir Oliver mean by an "educated person"? He has not told us. He might, of course, have so defined the term that his statement would be true by definition. He might, for example, have said that by "educated" he meant what the world means by it. In that case the indictment would be just. For the world has never deemed incompleteness of mathematical ignorance to be essential to education. If, however, Mr. Lodge wishes us to understand that by "educated" he means *liberally* educated, then the indictment is unjust, provided one conceives liberal education as the late Lord Kelvin conceived it; for this great man used to tell his students that among the "essentials of a liberal education is a mastery of Newton's 'Principia' and Herschel's 'Astronomy.'" But are there not educators who would deny Kelvin's contention? Undoubtedly there are and have been many of them. Such educators, for example, as Matthew Arnold, John Henry Newman and Thomas Huxley, widely divergent as are their outlooks upon the world, would yet unite in denying the contention impetuously or even with scorn.

It is evident that Mr. Lodge's deliverance is debatable. Certainly it is worthy of consideration. But who will consider it? May we expect its consideration by those whom it inculpates? If so, what should we expect the culprit to say? Well, he might speak as follows: I am an ordinary representative of the large class of average educated persons. You, Mr. Lodge, presumably represent the class of average mathematicians. You have said of me and my kind that our mathematical ignorance has always been complete and shameless. I in my turn desire to say of you and your kind that your average mathematician has always been completely and shamelessly indifferent in the matter of disclosing to his fellow men and women the cultural value of his science. Regarding our mathematical ignorance, which I regret to say is profound, though it is not quite complete nor entirely unashamed, I desire to say that it is not wholly due to the lack in us of mathematical faculty, but is due in large measure to the failure of mathematicians to show us that their science, over and above the appeal it makes to a class of specialists and technicians, is qualified to minister in any precious or important way to the spiritual needs which we have in common with all mankind.

Some such retort the average educated person may, I think, be conceived as making, not without justice, to Sir Oliver's allegation. In saying this I am far from intending to say that mathematicians have been wanting in devotion or patience or skill in presenting their science to multitudes of boys and girls and young men and women in its *technical* aspects. Nor do I mean to intimate that mathematicians may be rightly blamed for not making genuine mathematicians out of more than a very few of those they thus instruct, for every one knows that the genuine mathematician must be born before he can be bred. What I do mean is this: Among the countless host and endless variety of ideas that enter as components into the structure of mathematics there are a few concepts of so great generality and so great organizing power that they are superior to all the others in dignity and importance, serving as bases of the stately edifice or as its general framework or as central supporting pillars, like a tree-trunk to the tree or like a spinal column in the case of a vertebrate animal, giving the whole ideal architecture its substance, its character, its coherency, and its everlasting stability. Now, for the full unfolding of the implicit content of these supreme ideas it is indeed necessary to employ all the curious symbolism and all the other intricate machinery that more than twenty centuries of mathematical ingenuity have been able to invent, for these supreme ideas are just the things of which mathematics is the science. But—and herein lies the justice of the foregoing supposed retort—the great ideas themselves are all of them near at hand, and it is possible to present them, as mathematicians never have presented them, in their more obvious aspects and ruder outlines intelligibly to their fellow men and women; it is possible so to present them that the layman may understand better than he has ever understood what are the things that make up the subject-matter of the mathematician's silent meditations; it is possible so to present them that the science which Plato called "divine," which Goethe called "an organ of the inner higher sense," which Novalis called "the life of the gods," and which Sylvester called "the Music of Reason," shall not seem to laymen to be remote from their interests nor detached from reality, but that it shall appear to them to deal as it does deal with the very essence of reality, penetrating life in all its dimensions.

What are those great concepts?

There is room here merely to glance at a few of them, to call their names and to indicate some of their more obvious aspects somewhat as a traveler in the foothills may note the peaks of a great mountain range above and beyond.

Among the major mathematical ideas there can be little doubt, I think, that the concept to which Mr. Bertrand Russell has given the name of propositional function is supreme. Every one is more or less familiar with the notion of the lawful dependence of one or more variable things upon other variable things, as the area of a rectangle upon the lengths of its sides, as the distance traveled upon the rate of going, as the volume of a gas upon temperature and pressure, as the prosperity of a throat specialist upon the moisture of the climate, as the attraction of material particles upon their distance asunder, as the rate of chemical change upon the amount or the mass of the substance involved, and so on and on without end. This notion of mutual dependence and reciprocal determination which is thus exemplified in every turn and feature of life and the world and whose scientific name of function was first pronounced, it is said, by Leibnitz, is indeed a very powerful concept; it has played a dominant rôle in modern mathematical analysis, giving at once name and character to certain great branches, as the theory of functions of the real variable and the theory of functions of the complex variable. Yet this Leibnitzian concept, powerful as it is, is far inferior to that denoted by the term propositional function, which embraces the former merely as an exceedingly important special case. What, then, are we to understand by this more comprehensive term?

The answer is that a propositional function is any statement containing one or more real variables, where by a real variable is meant a name or a symbol whose meaning, or value as we say, is not determined by the statement, but to which we can at will assign in any order we please one or more values or meanings, now one and now another. I fear that what I have just said is too general and too abstract to be quite intelligible. The idea can be made clear, however, by some simple examples, provided the reader will understand that the examples are related to the general concept in question as a burning match to a world-conflagration or as a few water drops to a boundless ocean. Let us denote real variables by such italicized symbols as x, y, z, w, etc. Then for concrete and familiar examples of what is meant by propositional function we may cite the following quite at random: x is a man; x is a lover of y; x is the specific gravity of y; x, y and z are the coordinates of a point on the sphere whose center has x', y', z' for coordinates and whose radius is w; and so on *ad infinitum*. How many variables may enter a propositional function? As many as we please. How many such functions are there? It is evident that their name is legion—the host of them is literally infinite in multitude. Even so, you may wish to say, the examples are not impressive. And you may naturally doubt whether the concept they serve to exemplify can be so gigantic and majestic after all. I repeat, however, that the idea is sovereign. In the great and growing system of mathematical ideas, the concept of propositional function is indeed "like Jupiter among the Roman gods, first without a second." Its majesty, its power, its subtlety, the immeasurable range and depth of its significance can not be perceived and felt at once, but only more and more with days and months and years of reflection.

Let us reflect a little upon it. Every one knows that nothing can be more important than propositions. Why are propositions so important? Because truth and falsehood are so important, and propositions are just those living things in which truth and falsehood reside or to which they attach—a proposition is whatever is true or is false. Well, we are going to see presently that propositional functions are related to propositions as matrices are related to the things they mould. Again, no one can fail to recognize the importance of the idea of a class. Without this idea life would no doubt be possible but human life would not, not even for a day. So important is it that in all philosophic ages the concept of class has been held, not quite justly, to be supreme, and for more than two thousand years logic, the science of thought as thought, has been the doctrine of classes. What of it? As in the case of propositions, so here too: We shall see that it is from propositional functions that what we call classes derive their definitions and the determination of their content. And what shall we say of relations? Who does not know that our universe presents itself under scrutiny as an infinite plexus of relations? Who does not know that what we call things—whether they be objects of sense like the moon or objects of pure thought like the orbit of the moon—are but nodes or ganglia where relations meet and pass like a mesh of invisible wires uniting the many into one? "Being," says Lotze, "consists in relations." And it is not things themselves, says Henri Poincaré, that science can reach, as the naïve dogmatists think, but only the relations of things. "Outside of these relations there is no knowable reality." What, then, shall we think of propositional functions if these turn out, as in fact they do turn out, to be the forms in which all relations, whether of things or of ideas, are moulded and defined?

To see the connection of propositional functions with propositions it will suffice to consider some familiar propositional function, the simpler the better. Consider the homely function; x is a man. Observe that this function, though it has the form of a proposition, is not a proposition; for a proposition is true or is false, but the statement—x is a man—can be neither true nor false so long as x has not received an admissible meaning or value (such as Socrates, say), but when such a value has been assigned we no longer have a function, but have a proposition; namely, Socrates is a man. Thus we see how propositions, which are constant and which may be called values of the function, are derivable from the function, which in its turn is not a constant, but is a variable owing its variability to the presence in it of one or more unassigned terms or variables such as x, y, etc. Is this so easy as to be uninteresting and unimpressive? If so, that is no reason for being disheartened, for there are difficulties enough near at hand. Let us notice one of them. I have spoken of "admissible" values of x.

What is such a value? It is one that, when put for x in the function, yields a significant statement, a statement, that is, that makes sense, as we say. In other words, an admissible value of x is one that converts the function into a proposition, into something that is true or else false and not into mere nonsense. But our universe contains an infinitude of constant terms. Are all of these admissible values of x? No; the term John Smith is admissible, and so is the name Fido (the designation of my dog) for the statement—Fido is a man—is significant, it is a proposition, although it is false. Indeed, if it were not a proposition it could not be false, for, as already pointed out, propositions are the only things that can be false or true. Now men constitute a *class*. Is this class an admissible value of x? Evidently it is not, for the class is logically subsequent to the individuals composing it, and so it can not, without logical contradiction or nonsense, be said to be one of its own members. Accordingly, the statement—the class of men is a man—is neither true nor false; it is, rightly understood, just sheer nonsense. It is easy thus to see that our simple and homely propositional function, x is a man, cleaves the universe of terms or values into two infinite parts; one part being composed of inadmissibles and the other of admissibles. Is the line of cleavage always sharply defined? No; it may be doubtful whether a given term is or is not admissible, for we may ask, for example, whether the sweetness of sugar or the glory of renown is, in case of the function under consideration, an admissible value of x. There is here an open and inviting field for scientific research, the problem being to determine the best possible criteria for deciding, in case of any given propositional function, what terms or values are admissible and what ones are not. The situation may be likened to that of physical organisms, for there are plants and there are animals, but in the case of some living organisms there is at present no means of deciding to which division of the kingdom they belong. It is plain, too, that just as a propositional function containing a single variable parts the universe of terms into two infinite divisions of terms, so a function of *two* variables sunders the universe of *couples* of terms into two infinite divisions of couples; and so on and on for the case of three or of four or of n variables.

Again confining our attention to some concrete propositional function of a single variable, let us, for the sake of convenience, denote it by the symbol $F(x)$. Then, if a and b denote admissible values of x, $F(a)$ will be a proposition and so will $F(b)$.

It may happen that one of these will be true and the other one false. Such will be the case if, for example, $F(x)$ means "x was a great Italian poet" and if a denote Dante and b denote Shakespeare. Thus it appears not only that every propositional function of a single variable divides the universe of terms, as we have seen, into two grand divisions, but also—and this is exceedingly important—that the grand division composed of admissible values is at the same time separated by the function into two classes: namely, the class of values that yield true propositions and the class of those that yield false ones. Now the former of these classes, because of its relation to truth, enjoys the distinction of being regarded as *the* class determined by the given function. Is there a function that in this sense determines the other class? Yes; the other class is determined by the negative of the given function. Now a value of the variable that converts a function into a true proposition is said to *satisfy* the function. Accordingly a class consists of all and only the terms or values that satisfy some propositional function. To each function there thus corresponds a determinate class, one and but one. Is the converse true? A class being given, is it true that there is one and but one function that determines the class? Far from it. Given a class, there are, in general, many different functions, each of which suffices to determine it. Thus "x is a prime number" and "x is not divisible by any number except 1 and x" are two functions determining the same class, having, as we say, the same extension, the difference of the functions being what is called intensional difference. And this brings us to the weighty notion of *equivalence* among functions, two propositional functions being said to be equivalent when and only when they determine the same class. It is a very important and often a very difficult problem, when a function is given, to determine whether certain other functions are or are not equivalent to it. Is there a universal class? There is. Such a class may be defined by the function: x is identical with itself, or with x. But it must not be inferred that a universal class includes all things, for such an inference would lead quickly to logical contradiction, or nonsense. A class that included all things would have to include itself and such inclusion is logically impossible, it is nonsensical. There are classes of individuals, classes of classes, classes of classes of classes, and so on upward forever; so that classes and their corresponding functions constitute a summitless hierarchy of types or ranks—a subtle matter that can not be further pursued here, but which has to be faced and which has been faced bravely and with much profit to philosophy and science by Messrs. Whitehead and Russell in their magnificent "Principia Mathematica." Suffice it to say here that the notion of identity and that of universal class must be defined independently for each rank in the hierarchy or else extended from rank to rank by means of some more or less plausible assumption. To the universal class of any given rank there corresponds its negative: namely, the class determined by a function not satisfied by any value of the variable. Despite the fact of its containing no members, this so-called empty or null class is said to exist for the reason that there exists a propositional function determining it. And this strange class is very important because it lies at the basis of that curious integer which is known as *zero* and which is so indispensable to science and to the conduct of civilized life.

I fancy that the non-mathematical reader may wish to say: "What has all this to do with mathematics? I have always supposed that mathematics is the science that deals with number and space, and I quite fail to see any very obvious or close or significant connection between propositional functions on the one hand and the various kinds of number and of space configurations on the other." Very well, let us pause here a moment to exhibit such a connection. You doubtless think nothing can be more familiar than the numbers 1, 2, 3, etc., with which we count. What is it that we humans mean by the number 3 or the number 5, for example? There are probably not more than 2,000 people in the world who can answer that question. In answering it, I shall be relating a piece of the very latest scientific news. Consider the propositional functions: x is a finger of my right hand; y is a finger of my left hand. Each of these functions determines a class. The two classes, c and c', are said to be *equivalent* because we can pair the things of the one class with those of the other in a thing-to-thing way. By such a pairing we are said to *transform* each class into the other, and so we note in passing that the important mathematical term transformation does not mean what it means in general literature, for in general literature it involves the idea of transmutation; but classes that are mathematically transformed are merely associated and are not changed or transmuted into something else. Mathematical transformation is purely psychical, it is merely a lawful way of transferring our attention from a given thing to a definitely associated thing. To return from this digression: are there any other classes that are each equivalent to c and hence to c'? Obviously there are many of them, as the class c'' of letters in the word *write*, the class c''' of toes of a normal human foot, and so on and on. It is now essential to note carefully that there is a class C composed of all those equivalent classes. Observe that the members of C are not fingers, toes, letters, etc., but are classes c, c', etc. Do you know what C is called? In English it is called *Five* and is everywhere denoted by the symbol 5. Thus the number five is simply a certain class of certain equivalent classes, and the name five and the symbol 5 are simply the name and the symbol of that class. Exactly the like is true of all other integers. Does the fact seem strange? Well, science can not agree to discover nothing but what is familiar. Have we answered the question: what is an integer? No, we have merely indicated how to tell precisely what is meant by the number of any *given* class. To tell what is meant by the general phrase, an integer, we must go higher, we must form a more tenuous concept, we must say that the phrase stands for the class Σ of all the classes formable, like C, of classes of things. Here we gain an insight into the reason why the doctrine of number makes such severe demands upon intellection. For observe that a finger is indeed a sensible thing, but that the class c of fingers is not sensible, but is a pure concept. The class C is then a concept of concepts, and the class Σ is a concept of concepts of concepts; and accordingly the meaning of the phrase, an integer, is thrice removed from the domain of sense.

I have now shown how propositional functions are connected with the most familiar of mathematical things; namely, the integers, or count numbers of the shopkeeper. But there are many other sorts of number, as the rational fractions; as the real numbers including such as e, π, $\sqrt{2}$, etc.; as the complex numbers, which involve an even root of some negative real number; and so on. And you may wish to ask: Are all these and are all the configurations studied in geometry connected with propositional functions? They are. To show it, however, it is necessary to show, as I have promised to show, that the notion of propositional function is the source of the great concept of relations.

And that is not hard to do. Let $F(x,y)$ denote any one whatever of the propositional functions that contain two variables x and y. To fix our ideas, as French writers say, let us take the function: x is the lover of y. Any such function determines a class of *couples*, namely, the ensemble of ordered pairs of values of x and y that convert the function into a true proposition. If a loves b, then a and b, taken in the order named, are together one of the couples. Such a couple may be called an *element* of the *relation determined by the function.* Now just what, you may wish to ask, is the relation so determined? The answer is: the relation is the ensemble of all such elements or couples. Thence it appears that in mathematics a relation is regarded as consisting of its *extension* as distinguished from its *intension.* Thus the two functions, "x is greater than y," and "u is neither equal to nor less than v," though they differ in respect to intension, are said to determine the same relation because the two classes of couples determined by them are identical. Every propositional function of two or more variables determines a relation, and the relation is called dyadic or triadic or n-cornered, according as it is determined by a function of 2 or of 3 or of n variables. It is at once evident how infinitely rich and complicate the world of relations is. Let us for the present speak only of dyadic relations. If R denote such a relation, we may say that x has the relation to y by writing xRy. At once we see that dyadic relations have direction or sense, for if aRb, it is generally, though not always, false to say bRa. The things that can stand before a given relation constitute its *domain;* those that can stand after it, its *co-domain;* the domain and co-domain, which may or may not have things in common, together constitute the relation's *field.* Obviously relations present themselves under certain striking types. Thus there are *symmetric* relations, equality for example or diversity, such that if aRb, then also bRa; there are *asymmetric* relations, father or greater, for example, such that, if aRb, then never bRa; there are *non-symmetric* relations, friend, for example, or brother, such that, if aRb, then sometimes but not always bRa; there are *transitive* relations, less for example or identity, such that, if aRb, and bRc, then aRc; and there are intransitive relations, non-transitive relations, one-to-many relations, many-to-one relations, one-to-one relations, like that of husband or that of wife in well-regulated communities, many-to-many relations, and numerous other distinctions.

And now a word regarding applications to familiar mathematical things. A rational fraction is simply a kind of dyadic relation among integers. Thus, the fraction *two thirds* is the relation—that is, the class of couples (a, b)—determined by the propositional function, $3x = 2y$. Of real numbers as distinguished from rational numbers I shall speak presently in connection with the concept of *Limit.* As to geometry, any one a little acquainted with the analytical method discovered by Descartes and Fermat can readily see that any space configuration whatever is a relation. For, to pass abruptly to 3-cornered relations, if x, y, and z be the coordinates of a point—that is, if they be its distances from three chosen planes of reference—then any propositional function, such as $2x^2+4y=9z+3$, determines a relation among points and this relation is called a surface.

In support of my statement that the notion of propositional function is sovereign among mathematical ideas, I have said enough in this sketch to show that this omnipresent notion embraces the great concepts of proposition, class, and relation like an infinite envelope inwrapping them completely and touching them, so to speak, at every point. I must now hasten on to other pillar-ideas without, however, passing beyond the range of "Jupiter," for that can not be done.

A masterful idea that owes its precision and its great fame to mathematics but which, as we shall see, has everywhere penetrated, under a thinner or thicker disguise, the history of thought and aspiration, is the notion of limit. May I remind you by an example what the notion means? Suppose we are operating in the field of the *rational* numbers. Consider the series or sequence of all the rationals such that the *square* of each is less than, say, the sacred number 7. Then we say that the sequence has a *limit*, which we call the square root of 7 and denote by $\sqrt{7}$. But this thing, this limit, is not a rational number; it is something outside the field of rationals; it is merely indicated and approached by the rational sequence. In relation to our field of operation, this limit is then not an actual thing, but is purely ideal and the process of approaching it along the sequence is, as you see, a process of idealization. It is thus evident that the notion of limit and the process of limits, which lie at the basis of the Newtonian and Leibnitzian calculus and which are indispensable to mathematical computation and generalization (as leading, for example, to the concept of such irrational numbers as $\sqrt{7}$), it is evident, I say, that this concept and this process are but mathematicized forms of those ideals and that process of idealization which in other fields of interest have given man his dreams of perfection, whether in ethics or in religion, or in art, or in governance, or in knowledge. Every manner of perfection, every genuine ideal, every source of supernal light upon our human pathways, is indeed some great unmathematicized limit, unattainable indeed, yet indicated and pursued by familiar sequences of experience in our common life.

In our hasting excursion among the great mathematical ideas, we must not fail to glance at the concept called a system of postulates. It is a system composed of a few so-called axioms or assumptions or propositions called primitive because they are taken for granted, it being impossible to prove everything. The purpose or office of such a system is to serve as a foundation for a doctrine all of whose propositions, except the postulates themselves, are to be logically demonstrated. If a postulate system is to be an ideal one it must be such that the postulates are compatible—that is, not mutually contradictory—and they must be independent in the sense that none of them can be logically derived from the rest. In the course of more than two thousand years, and especially in our own day, numerous such systems, or mathematical branch-foundations, have been discovered. A famous one of these is found in the late Professor Hilbert's "Foundations of Geometry." In every postulate system the postulates are statements about certain terms, or elements as they are often called. These terms are *not defined* beyond the requirement that they must satisfy the postulates or, in other words, that they must be things about which the postulates make true statements. In Hilbert's system, for example, the undefined terms are point, line, and plane. Since the terms are undefined we may as well replace them in the postulates by the variables x, y and z and then it appears on the very face of the postulates, since they now talk about the variables, x, y, and z, that they are not propositions, but are propositional functions. And hence it appears that the so-called doctrine erected upon them is really not a doctrine, for a doctrine must be true or false, consisting of propositions, but is really a *doctrinal function* depending upon the propositional functions at its base. By giving these variables admissible meanings, or values, we get doctrines from the doctrinal function just as propositions are obtained from propositional functions. One of the impressive facts recently discovered in this field is that from a given doctrinal function we can thus derive an infinitude of doctrines, some of them true, some of them false. Inasmuch as these have the same foundation, they are all of them of the same form; they are *isomorphic,* as we say; they are logically *one,* but psychologically they are *infinitely many.* Who can tell the disadvantages that would attach to living in a world, if there were such a world, where every doctrine required a foundation of its own?

Before quitting the subject of postulate systems I desire to mention two considerations, one of them touching the humanity of mathematics, the other indicating one of its fundamental bearings on philosophy. The first consideration is that in seeking a postulate system to serve as the support of a mathematical branch, the mathematician is engaged in the very human work of searching for principles, for beginnings that will guide, and his activity, though it is distinguished by its precision and ideality, is, in point of *kind,* not different from that common quest of man in all ages for fundamental truths which has eventuated, not merely in such scientific things as the principles of Newtonian mechanics, for example, but in decalogues, in creeds, in political constitutions, and in principles of jurisprudence. The other consideration is that by their postulational research, mathematicians have conclusively demonstrated that the now age-long attempt of philosophy to derive the universe from a principle or from a consistent set of principles can never be successful, not because man is lacking in wisdom but because the problem admits of no solution. How is this shown? It is shown by this: Geometricians have discovered three geometries, one of them called Euclidean because, like Euclid, it postulates that in a plane there is through a given point one and but one line parallel to a given line; one of them called Lobachevskian because, like Lobachevsky, it postulates more than one such parallel; and one of them called Riemannian because Riemann postulated that there should be no such parallels at all. Now each of these three classic geometries is internally consistent and is, therefore, indestructible. But the geometries contradict one another. Accordingly we have in our universe these three eternal but mutually incompatible doctrines. If a consistent theory of the universe could be constructed on the basis of a single set of compatible postulates, then the geometries in question, being a part of the universe, would have to be derived with all the rest of it as harmonious affairs; but this can not happen, since neither men nor gods can render concordant two things that contradict each other categorically. To think otherwise would be to abolish the very notion of logical harmony. Herewith, then, is established by mathematics an eternal limit to the possible advancement of philosophical speculation.

Space fails me to deal suitably with such momentous concepts as those of infinity, group, hyperspace, and invariant. As to the first of these I may perhaps be permitted to refer an interested reader to my book, "The New Infinite and the Old Theology," where I have presented, in the language current among educated people, the mathematical concept of infinity together with its bearings upon some problems of rational theology. The concept of group, which entered mathematics about a century ago and which, besides giving rise to an extensive doctrine of its own, has come to serve more and more for the characterization and classification of other mathematical branches, would require a separate essay to present it adequately to laymen along with its bearings upon general thought, ancient and modern. The same must be said of hyperspace.

As to the notion of invariance, it has played so great a rôle, not only in mathematics, but in every cardinal field of human interest, that I can not close without giving at least a little sketch of its nature and significance. What is an invariant? Broadly speaking, it is anything, simple or complex, that remains unaltered when other things connected with it suffer change. The mathematical theory of invariance is about as old as our American independence. Its beginning was like a mustard seed. The seed was an observation by Lagrange that the discriminant, $b^2 - ac$, of the quadratic equation, $ax^2 + 2bxy + cy^2 = 0$, is the same for all the countless equations that can be obtained from the given one by replacing the x in it by $x + \lambda y$, λ being allowed to take any and all numerical values. The mentioned replacement is a very simple example of what is known in mathematics as a transformation, of which I spoke briefly above. Thus what Lagrange noticed was the fact that the above-mentioned discriminant remains invariant under an endless number of transformations. If the reader will take the trouble to reflect for a moment upon the fact that an equation may contain any given number whatever of variables and upon the further fact that, the number of variables being assigned, the equation involving them can have any degree whatever, and if he will then reflect that the number of coefficients increases very rapidly with the number of variables in the equation and with its degree, he can not fail to glimpse the magnitude of the problem which consists in searching out *all* those combinations of the coefficients or of the coefficients and the variables together that remain unchanged when the equations are transformed by replacing in them, not merely one of the variables, as in the example of Lagrange, but each of them by an expression containing them all. That is the magnificent enterprise to which, as a result of Lagrange's tiny observation, mathematicians engaged for some generations, first Gauss, then George Boole, then Arthur Cayley and James Joseph Sylvester and then a small army of master-workmen both British and Continental. The event is a stately doctrine variously styled the theory of invariants and covariants, the theory of quantics, as Cayley was wont to name it, and the calculus of forms, as it was more poetically conceived by Sylvester. The notion of invariant has been extended far beyond the range of algebra in which it originated, into all branches not only of mathematics, but of natural science. A little reflection will suffice to show that nothing can be closer to the heart of man than this seemingly cold and arid mathematical concernment with the doctrine of invariant forms. For it is obviously only the mathematical aspect of man's quest, in all times and places of our fluctuant world, for abiding reality and which in art has given us the doctrine of eternal archetypes of beauty, in jurisprudence the ancient conception of *lex naturæ,* in science the idea of indestructible atoms and of invariant natural order or law, and in religion and theology such dreams as an immutable God and an immortality for human souls.

Finally a word respecting the bearings of mathematics upon ethics. No one can contemplate the ideal cosmos disclosed in mathematics, no one can realize how indissolubly ideas there are interlocked, no one can perceive that there consequences follow from chosen beginnings with a fatality against which not even God, said Plato, can contend, no one, I say, can face such aspects of our world without having his ethical sense touched by a sobering awe.